Effective Field Theory for Cold Atoms

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Abstract. Effective Field Theory (EFT) provides a powerful framework that exploits a separation of scales in physical systems to perform systematically improvable, model-independent calculations. Particularly interesting are few-body systems with short-range interactions and large two-body scattering length. Such systems display remarkable universal features. In systems with more than two particles, a three-body force with limit cycle behavior is required for consistent renormalization already at leading order. We will review this EFT and some of its applications in the physics of cold atoms. Recent extensions of this approach to the four-body system and N-boson droplets in two spatial dimensions will also be discussed.

INTRODUCTION

The Effective Field Theory (EFT) approach provides a powerful framework that exploits the separation of scales in physical systems [1]. Only low-energy (or long-range) degrees of freedom are included explicitly, with the rest parametrized in terms of the most general local contact interactions. This procedure exploits the fact that a low-energy probe of momentum k cannot resolve structures on scales smaller than $1/k$. (Note that $\hbar = 1$ in this talk.) Using renormalization, the influence of short-distance physics on low-energy observables is captured in a few low-energy constants. Thus, the EFT describes universal low-energy physics independent of detailed assumptions about the short-distance dynamics. All physical observables can be described in a controlled expansion in powers of kl, where l is the characteristic low-energy length scale of the system. The size of l depends on the system under consideration: for a finite range potential, e.g., it is given by the range of the potential. For the systems discussed here, l is of the order of the effective range r_e.

We will focus on applications of EFT to few-body systems with large S-wave scattering length $|a| \gg l$. For a generic system, the scattering length is of the same order of magnitude as the low-energy length scale l. Only a very specific choice of the parameters in the underlying theory (a so-called *fine tuning*) will generate a large scattering length a. Nevertheless, systems with large scattering length can be found in many areas of physics. The fine tuning can be accidental or it can be controlled by an external parameter. Examples for an accidental fine tuning are the S-wave scattering of nucleons or of ^{4}He atoms. For alkali atoms close to a Feshbach resonance, a can be tuned experimentally by adjusting an external magnetic field. At very low energies these systems behave similarly and show universal properties associated with the large scattering length.

We will start with a brief review of the EFT for few-body systems with large a and then discuss some of the universal properties predicted in three and two spatial dimensions.

THREE-BODY SYSTEM WITH LARGE SCATTERING LENGTH

We consider three spatial dimensions first. For typical momenta $k \sim 1/|a|$, the EFT expansion is in powers of $l/|a|$; in leading order we can set $l = 0$. We start with a two-body system of nonrelativistic bosonic atoms with large S-wave scattering length a and mass m. At sufficiently low energies, the most general Lagrangian may be written as:

$$\mathscr{L} = \psi^\dagger \left(i\partial_t + \frac{\vec{\nabla}^2}{2m} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 - \frac{D_0}{6} (\psi^\dagger \psi)^3 + \dots, \tag{1}$$

where the C_0 and D_0 are nonderivative two- and three-body interaction terms, respectively. The strength of the C_0 term is determined by the scattering length a, while D_0 depends on a three-body parameter to be introduced below. The dots represent higher-order derivative terms which are suppressed at low-energies. For momenta k of the order of the inverse scattering length $1/|a|$, the problem is nonperturbative in ka. The exact two-body scattering amplitude can be obtained analytically by summing the so-called *bubble diagrams* with the C_0 interaction term. The D_0 term does not contribute to two-body observables. After renormalization, the resulting amplitude reproduces the leading order of the well-known effective range expansion for the atom-atom scattering amplitude: $f_{AA}(k) = (-1/a - ik)^{-1}$, where the total energy is $E = k^2/m$. If $a > 0$, f_{AA} has a pole at $k = i/a$ corresponding to a shallow dimer with binding energy $B_2 = 1/(ma^2)$. Higher-order derivative interactions are perturbative and give the momentum-dependent terms in the effective range expansion.

We now turn to the three-body system. Here, it is useful to introduce an auxiliary field for the two-atom state (see Ref. [2] for details). At leading order, the atom-dimer scattering amplitude is given by the integral equation shown in Fig. 1. A solid
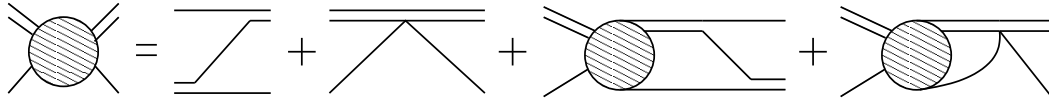

FIGURE 1. Integral equation for the atom-dimer scattering amplitude. Single (double) line indicates single atom (two-atom state).

line indicates a single atom and a double line indicates an interacting two-atom state (including rescattering corrections). The integral equation contains contributions from both the two-body and the three-body interaction terms. The inhomogeneous term is given by the first two diagrams on the right-hand side: the one-atom exchange diagram and the three-body interaction. The integral equation simply sums these diagrams to all orders. After projecting onto S-waves, we obtain the equation

$$\mathscr{T}(k,p;E) = \frac{16}{3a} M(k,p;E) + \frac{4}{\pi} \int_0^\Lambda \frac{dq\, q^2 M(q,p;E)}{-1/a + \sqrt{3q^2/4 - mE - i\varepsilon}} \mathscr{T}(k,q;E), \tag{2}$$

for the off-shell atom-dimer scattering amplitude with the inhomogeneous term

$$M(k,p;E) = \frac{1}{2pk} \ln\left(\frac{p^2 + pk + k^2 - mE}{p^2 - pk + k^2 - mE} \right) + \frac{H(\Lambda)}{\Lambda^2}. \tag{3}$$

The logarithmic term is the S-wave projected one-atom exchange, while the term proportional to $H(\Lambda)$ comes from the three-body force. The S-wave atom-dimer scattering amplitude $f_{AD}(k) = [k\cot\delta_0 - ik]^{-1}$ is given by the solution $\mathscr{T}$ evaluated at the on-shell point: $f_{AD}(k) = \mathscr{T}(k,k;E)$ where $mE = 3k^2/4 - 1/a^2$. The three-body binding energies B_3 are given by those values of $E < 0$ for which the homogeneous version of Eq. (2) has a nontrivial solution.

Note that an ultraviolet cutoff Λ has been introduced in (2). This cutoff is required to insure that Eq. (2) has a unique solution. All physical observables, however, must be independent of Λ, which determines the behavior of H as a function of Λ [2]:

$$H(\Lambda) = \frac{\cos[s_0 \ln(\Lambda/\Lambda_*) + \arctan s_0]}{\cos[s_0 \ln(\Lambda/\Lambda_*) - \arctan s_0]}, \qquad (4)$$

where $s_0 = 1.00624$ is a transcendental number and Λ_* is a three-body parameter introduced by dimensional transmutation. This parameter cannot be predicted by the EFT and must be determined from a three-body observable. Note also that $H(\Lambda)$ is periodic and runs on a limit cycle. When Λ is increased by a factor of $\exp(\pi/s_0) \approx 22.7$, $H(\Lambda)$ returns to its original value.

In summary, two parameters are required to specify a three-body system at leading order in $l/|a|$: they may be chosen as the scattering length a (or equivalently B_2 if $a > 0$) and the three-body parameter Λ_* [2].

UNIVERSAL PROPERTIES IN 3D

This EFT confirms and extends the universal predictions for the three-body system first derived by Efimov [3]. The best known example is the *Efimov effect*, the accumulation of infinitely many three-body bound states at threshold as $a \to \pm\infty$. Universality also constrains three-body scattering observables. The atom-dimer scattering length, e.g., can be expressed in terms of a and Λ_* as [3, 4]

$$a_{12} = a\left(1.46 - 2.15\tan[s_0\ln(a\Lambda_*) + 0.09]\right)\left(1 + \mathscr{O}(l/a)\right), \qquad a > 0. \quad (5)$$

Note that the log-periodic dependence of H on Λ_* is not an artefact of the renormalization and also shows up in observables like a_{12}. This dependence could, e.g., be tested experimentally with atoms close to a Feshbach resonance by varying a. Similar expressions can be obtained for other three-body observables such as scattering phase shifts as well as three-body recombination and dimer relaxation rates [4, 5].

Since up to corrections of order $l/|a|$, low-energy three-body observables depend on a and Λ_* only, they obey non-trivial scaling relations. If dimensionless combinations of such observables are plotted against each other, the must fall close to a line parametrized by Λ_* [2, 4]. An example of a scaling function relating ^{4}He trimer ground and excited state energies $B_3^{(0)}$ and $B_3^{(1)}$ is shown in the left panel of Fig. 2. (A related scaling function was obtained in Ref. [6].) The data points show calculations using various approaches and ^{4}He potentials. Since different potentials have approximately the same scattering length but include different short-distance physics, different points on this
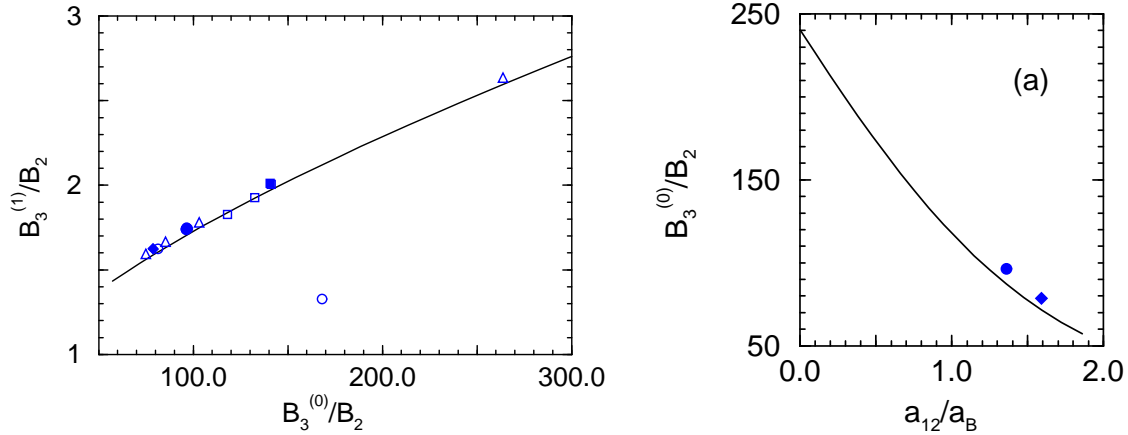
FIGURE 2. Scaling functions relating the ^{4}He trimer ground and excited state energies (left panel) and the ^{4}He trimer ground state and the atom-dimer scattering length (right panel). The data points are calculations using various methods and ^{4}He potentials (see Ref. [7] and references therein).

line correspond to different values of Λ_*. The small deviations of the potential model calculations are mainly due to effective range effects. They are of the order $r_e/a \approx 10\%$ and can be calculated at next-to-leading order in EFT. The calculation corresponding to the data point far off the universal curve can easily be identified as problematic since the deviation from universality by far exceeds the expected 10%. The right panel shows the scaling function relating the ^{4}He trimer ground state energy $B_3^{(0)}$ and the atom-dimer scattering length a_{12}. (Note that $a_B \equiv 1/\sqrt{mB_2}$.) A similar scaling relation is observed in nuclear physics between the spin-doublet neutron-deuteron scattering length and the triton binding energy and is known as the Phillips line.

Recently, we have extended the effective theory for large scattering length to the four-body system [8]. It is advantageous in this case to use an effective quantum mechanics framework, since one can directly start from the well-know Yakubovsky equations. We have shown that no four-body parameter enters at leading order in $l/|a|$. Therefore renormalization of the three-body system automatically guarantees renormalization of the four-body system. As a consequence, there are universal scaling functions relating three- and four-body observables as well. As an example, we show the correlation between the trimer and tetramer binding energies in Fig. 3. The left panel shows the correlation between the ground state energies while the right panel shows the correlation between the excited state energies. The data points are calculations using various methods and ^{4}He potentials (see Ref. [9] and references therein). We conclude that universality is well satisfied in the three- and four-body systems of ^{4}He atoms.

The existence of these scaling functions is a universal feature of systems with large scattering length and is independent of the details of the short-distance physics. Similar correlations between few-body observables appear for example also in nuclear systems. The correlation between the triton and α particle binding energies (the Tjon line) can be explained using the same effective theory [10].
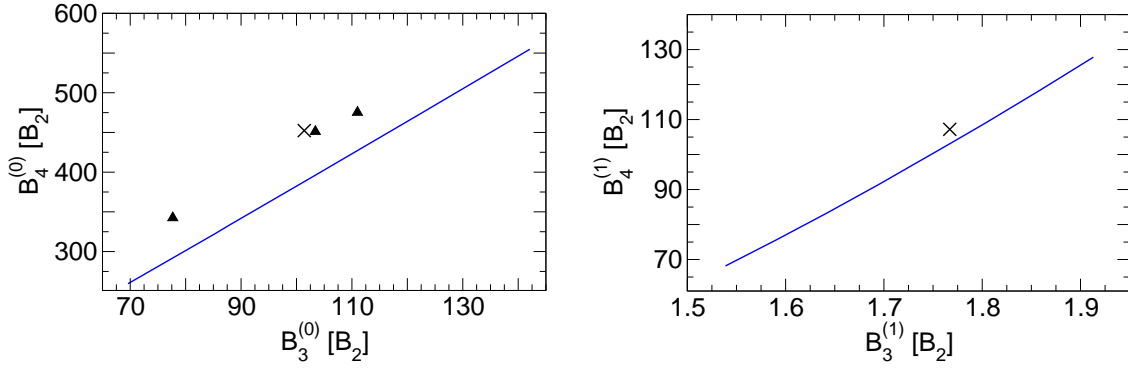
FIGURE 3. Scaling function relating the ^{4}He trimer and tetramer ground state (left panel) and excited state (right panel) energies. The data points are calculations using various methods and ^{4}He potentials.

N-BOSON DROPLETS IN 2D

In this section, we consider the universal properties of weakly interacting bosons with large scattering length (or equivalently a shallow dimer state) in two spatial dimensions (2D) [11]. In particular, we consider self-bound droplets of $N(\gg 1)$ bosons interacting weakly via an *attractive*, short-ranged pair potential. Our analysis relies strongly on the property of asymptotic freedom of 2D bosons with an attractive interaction.

In 2D, any attractive potential has at least one bound state. For the potential $-g\delta^2(r)$ with small g, there is exactly one bound state with an exponentially small binding energy, $B_2 \sim \Lambda^2 \exp(-4\pi/g)$, where Λ is the ultraviolet momentum cutoff (which is the inverse of the range of the potential). Asymptotic freedom provides an elegant way to understand this result. In 2D nonrelativistic theory, the four-boson interaction $g(\psi^\dagger\psi)^2$ is marginal. The coupling runs logarithmically with the length scale R, and the running can be found by performing the standard renormalization group (RG) procedure. For $g > 0$, the coupling grows in the infrared, in a manner similar to the QCD coupling. The dependence of the coupling on the length scale R is given by

$$g(R) = \left[\frac{1}{g} - \frac{1}{4\pi}\ln(\Lambda^2 R^2) \right]^{-1}, \tag{6}$$

so the coupling becomes large when R is comparable to the size of the two body bound state $B_2^{-1/2}$. This is in essence the phenomenon of dimensional transmutation: a dynamical scale is generated by the coupling constant and the cutoff scale. It is natural, then, that B_2 is the only physical energy scale in the problem: the binding energy of three-particle, four-particle, etc. bound states are proportional to B_2. In contrast to three spatial dimensions, there is no Efimov effect (or Thomas collapse) and no three-body parameter is required. The N-particle binding energy B_N, however, can be very different from B_2 if N is parametrically large. We use the variational method to calculate the size of the bound state. For a cluster of a large number of bosons, one can apply classical field theory. We thus have to minimize the expectation value of the Hamiltonian with respect to all field configurations $\psi(r)$ satisfying the constraint $N = \int d^2r\,\psi^\dagger\psi$. This is equivalent to a Hartree calculation with the running coupling constant $g(R)$ instead of

the bare one. In the limit of a large number N of particles in the droplet, some exact predictions can be obtained [11].

The system possesses surprising universal properties. Namely, if one denotes the size of the N-body droplet as R_N, then at large N and in the limit of zero range of the interaction potential:

$$R_{N+1}/R_N \approx 0.3417, \qquad N \gg 1. \tag{7}$$

The size of the bound state decreases exponentially with N: adding a boson into an existing N-boson droplet reduces the size of the droplet by almost a factor of three. Correspondingly, the binding energy of N bosons B_N increases exponentially with N:

$$B_{N+1}/B_N \approx 8.567, \qquad N \gg 1. \tag{8}$$

This implies that the energy required to remove one particle from a N-body bound state (the analog of the nucleon separation energy for nuclei) is about 88% of the total binding energy. This is in contrast to most other physical systems, where separating one particle costs much less energy than the total binding energy, provided the number of particles in the bound state is large. The $1/N$-corrections to Eqs. (7, 8) are calculable.

For the universal predictions (7, 8) to apply in realistic systems with finite-range interactions, the N-body bound states need to be sufficiently shallow and hence have a size R_N large compared to the natural low-energy length scale l. Depending on the physical system, l can be the van der Waals length l_{vdW}, the range of the potential or some other scale. As a consequence, Eqs. (7, 8) are valid in such systems for N large, but below a critical value,

$$1 \ll N \ll N_{\text{crit}} \approx 0.931 \ln(R_2/l) + \mathcal{O}(1). \tag{9}$$

At $N = N_{\text{crit}}$ the size of the droplet is comparable to l and universality is lost. If there is an exponentially large separation between R_2 and l, then N_{crit} is much larger than one and the condition (9) can be satisfied.

We can compare our prediction with exact few-body calculations for $N = 3, 4$. While the $1/N$-corrections are expected to be relatively large in this case, we can estimate how the universal result (8) is approached. The three-body system for a zero-range potential in 2D has exactly two bound states: the ground state with $B_3^{(0)} = 16.522688(1) B_2$ and one excited state with $B_3^{(1)} = 1.2704091(1) B_2$ [12, 13, 11]. Similarly, the four-body system for a zero-range potential in 2D has two bound states: the ground state with $B_4^{(0)} = 197.3(1) B_2$ and one excited state with $B_4^{(1)} = 25.5(1) B_2$ [14]. The prediction (8) applies to the ground state energies $B_3^{(0)}$ and $B_4^{(0)}$. The ratio $B_3^{(0)}/B_2 \approx 16.5$ is almost twice as large as the asymptotic value (8), while the ratio $B_3^{(0)}/B_4^{(0)} \approx 11.9$ is already considerably closer. These deviations are expected for such small values of N. Note, however, that the ratio of the root mean square radii of the two- and three-body wave functions is 0.306 [13], close to the asymptotic value (7).

It would be interesting to test the universal predictions (7, 8) both theoretically and experimentally for $N > 4$. On the theoretical side, Monte Carlo techniques appear to be a promising avenue. Furthermore, the experimental realizablity of self-bound 2D boson systems with weak interactions should be investigated.

SUMMARY & OUTLOOK

We have discussed the EFT for few-body systems with short-range interactions and large scattering length a. The main focus was on applications to cold atoms.

The renormalization of the three-body system with large a in three spatial dimensions requires a one-parameter three-body force governed by a limit cycle already at leading order in the expansion in $l/|a|$. As a consequence, two parameters are required to specify a three-body system: the scattering length a (or the dimer binding energy B_2) and the three-body parameter Λ_*. However, once these two parameters are given the properties of the three- and four-body systems are fully determined at leading order in $l/|a|$.

The large scattering length leads to universal properties independent of the short-distance dynamics. In particular, we have discussed universal expressions for three-body observables and universal scaling functions relating different few-body observables.

In two spatial dimensions, the three-body parameter Λ_* does not enter at leading order in the expansion in $l/|a|$ and N-body binding energies only depend on B_2. The asymptotic freedom of non-relativistic bosons with attractive interactions in 2D leads to some remarkable universal properties of N-body droplets.

Future challenges include the application to Halo systems in nuclear physics [15], and the possibility of coexisting condensates of atoms, dimers, and trimers in atomic physics [16]. The three-body effects discussed here will also become relevant in Fermi systems with three or more spin states.

ACKNOWLEDGMENTS

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